

Final Exam: Discrete Math, MTH 213, Fall 2017

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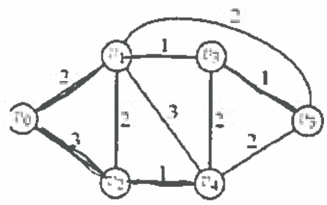
Score = $\frac{86}{86}$

Excellent +++

M. Said = perfect score on Exam I + Exam II + Final.

Apply Dijkstra's algorithm

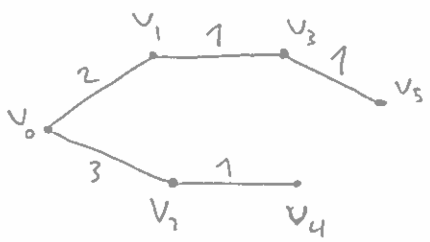


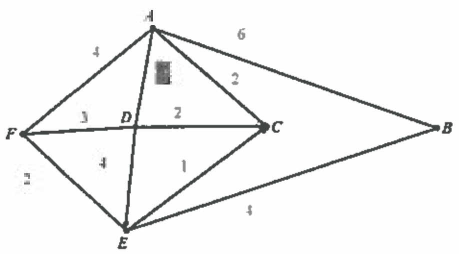
QUESTION 1. (8 points) Use DIJ-Algorithm to find the shortest path from V_0 to every other vertex.

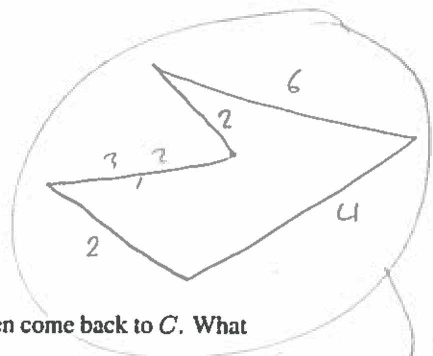
$V(s)$	$adj(V(s))$	$L(adj)$
$\{V_0\}$	V_1, V_2	$\boxed{L(V_1)=2}$, $L(V_2)=3$
$\{V_0, V_1\}$	V_2, V_4, V_3, V_5	$\boxed{L(V_2)=3}$, $L(V_3)=3$, $L(V_4)=5$, $L(V_5)=4$
$\{V_0, V_1, V_2\}$	V_3, V_4, V_5	$\boxed{L(V_3)=3}$, $L(V_4)=4$, $L(V_5)=4$
$\{V_0, V_1, V_2, V_3\}$	V_4, V_5	$\boxed{L(V_4)=4}$, $L(V_5)=4$
$V_0, V_1, V_2, V_3, V_4\}$	V_5	$\boxed{L(V_5)=4}$
$\{V_0, V_1, V_2, V_3, V_4, V_5$		

$E(s)$

$\{V_0 - V_1, V_0 - V_2, V_1 - V_3, V_2 - V_4, V_3, V_5\}$







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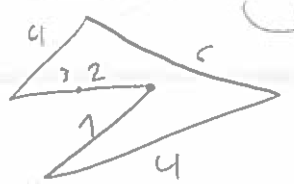
QUESTION 2. (5 points)

Assume that the post-office is at C. The mail-man must visit each block exactly once and then come back to C. What is the shortest trail that he must use?

$C \overset{2}{-} A \overset{6}{-} B \overset{4}{-} E \overset{2}{-} F \overset{3}{-} D \overset{2}{-} C = 19$



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QUESTION 3. (6 points) Consider the following Algorithm segment. Find the exact number of additions, multiplications, and subtractions that will be performed when the algorithm is executed. Then find the order of the Algorithm segment.

$m = 7; s = 0$

For $k := 3$ *to* $\lceil \frac{n}{2} \rceil$

For $i := 1$ *to* k

$s = s + m^2 + i - k$

next i

$L = k + s - 6$

next k

(1) For n is even $\lceil \frac{n}{2} \rceil = \frac{n}{2}$

$\Rightarrow$

$k = 3$ $\qquad k = \frac{n}{2}$

$3 \times 4 + 2$ $\qquad \frac{n}{2} \times 4 + 2$

$\#$ of all operations $= \dfrac{\frac{n-4}{2}(14 + 2n + 2)}{2}$

$\Rightarrow \dfrac{14n - 56 + 2n^2 - 8n + 2n - 8}{4}$

$= \dfrac{2n^2 + 8n - 64}{4} = \boxed{\frac{1}{2}n^2 + 2n - 16}$

$\boxed{\text{of order } \Theta(n^2), O(n^2)}$

(2) if n is odd $\Rightarrow \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$

$k = 3$ $\qquad k = \frac{n+1}{2}$

$3 \times 4 + 2$ $\qquad \frac{n+1}{2} \times 4 + 2$

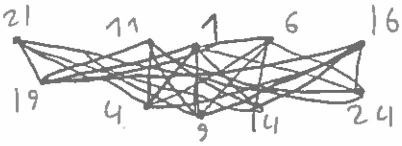
$\#$ of all operations $= \dfrac{\frac{n-3}{2}(14 + 2n + 4)}{2}$

$= \boxed{\frac{1}{2}n^2 + 3n - \frac{27}{2}}$

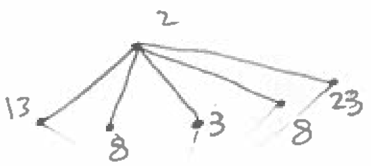
of order $\Theta(n^2), O(n^2)$

QUESTION 4. (6 points) Let D be a graph with vertex-set = $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, 16, 18, 19, 21, 23, 24\}$. Two vertices, say a, b, are connected by an edge iff $5 \mid (a + b)$. By drawing the graph, convince me that D consists of two components: one component is a complete bipartite graph and the other component is a tree.

5: 1-4, 2-3,
10: 1-9, 2-8, 3-7, 4-6
15: 1-14, 2-13, 4-11, 6-9, 7-8
20: 1-19, 2-18, 4-16, 6-14, 7-13, 9-11
25: 1-24, 2-23, 4-21, 6-19, 7-18, 9-16, 11-14
30: 6-24, 7-23, 9-21, 11-19, 14-16
35: 11-24, 14-21, 16-19,
40: 16-24, 19-21
45: 21-24



$K_{5,5}$



Tree of order 6

It's a connected Acyclic graph

QUESTION 5. (6 points) Convince me that $\neg[S_1 \to (\neg S_2 \to S_3)] \equiv [S_1 \wedge (\neg S_2 \wedge \neg S_3)]$

S_1	S_2	S_3	$\neg S_2 \to S_3$	$\neg S_2 \wedge \neg S_3$	$\neg(S_1 \to (\neg S_2 \to S_3))$	$S_1 \wedge (\neg S_2 \wedge \neg S_3)$
0	0	0	0	1	0	0
0	0	1	1	0	0	0
0	1	0	1	0	0	0
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	1	0	0	0
1	1	0	1	0	0	0
1	1	1	1	0	0	0

identical

$\Rightarrow \neg[S_1 \to (\neg S_2 \to S_3)] \equiv [S_1 \wedge (\neg S_2 \wedge \neg S_3)]$

QUESTION 6. (6 points) Use math induction to prove that $\sum_i^n i2^i = (n-1)2^{n+1} + 2$ for all $n \geq 1$

①: prove it for $n=1 \Rightarrow \sum_i^1 i2^i \Rightarrow 2 = 2$

②: Assume it's true for $n=k$, $k \geq 1 \Rightarrow \sum_i^k i2^i = (k-1)2^{k+1}+2$

③ prove it for $k+1 \Rightarrow$ prove: $\sum_i^{k+1} i2^i = k2^{k+2} + 2$:

$$\Rightarrow \sum_i^{k+1} i2^i = \underbrace{\sum_i^k i2^i}_{(k-1)2^{k+1}+2} + (k+1)2^{k+1} \Rightarrow 2^{k+1}(k+1+k-1)+2 = 2^{k+1}(2k)+2 = k2^{k+2}+2$$

$$\Rightarrow \boxed{\sum_i^{k+1} i2^i = k2^{k+2}+2}$$

QUESTION 7. (6 points) There are 100 baskets of apples. Each basket contains no more than 26 apples. Then

(i) There are at least n baskets containing the same number of apples. What is the maximum value of n that we all are sure about?

by pigeonhole principle: D $\{1, \ldots, 100\} \xrightarrow{\text{onto}}$ R $\{1, \ldots, 26\}$ $\Rightarrow n = \left\lceil \frac{100}{26} \right\rceil = 4$

(ii) There must exist a basket such that no more than m apples are in the basket. What is the minimum value of m that we all are sure about?

$m = \left\lfloor \frac{100}{26} \right\rfloor = 3$

QUESTION 8. (4 points) We have 7 holes labeled from 1 to 7 and we have 5 balls (red, blue, green, yellow, black). We need to put each ball in one hole. Assume that the red ball, green ball, and blue ball each must be placed in one of the holes labeled from 1 to 4 where the remaining balls each must be placed in one of the holes labeled from 5 to 7. In how many ways can we place the balls in the given holes?

4P3 x 3P2

QUESTION 9. (4 points) How many 3-digit odd numbers greater that 500 can be formed using the digits (2, 3, 4, 5, 6, and 7).

$X = X_1 X_2 X_3$

3C1 x 6C1 x 3C1 = 54

QUESTION 10. (4 points) There are 8 dots randomly placed on a circle. How many triangles can be formed within the circle (i.e., inside the circle)?

8C3 = 56

QUESTION 11. (4 points) 60 people at a party shake hands once with everyone else in the party. How many handshakes took place?

$60C2 = 1770$

QUESTION 13. (6 points) Let X be the number of people in the AUS-auditorium at some event. We know that $X < 130$. Also we know that $X \equiv 1 \ (mod\ 10)$ and $X \equiv 7 \ (mod\ 13)$. Find X.

$X \equiv 1 \ (mod_{10})$

$X \equiv 7 \ (mod_{13})$

$gcd(10, 13) = 1 \Rightarrow$ we can apply CRT $\Rightarrow$

$y_1 = [13 \ (mod_{10})]^{-1} = (3)^{-1} = 7$

$y_2 = [10 \ (mod_{13})]^{-1} = (10)^{-1} = 4$

$\Rightarrow X = [1 \times 13 \times 7 + 7 \times 10 \times 4]_{mod_{130}} = \boxed{111}$

14
27
37

QUESTION 14. (3 points) Write down T or F

(i) If $\exists x \in R$ such that $x^2 + 4 = 8$, then $x^2 + 1 = 5$ T

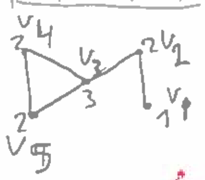
(ii) If $\exists x \in N^*$ such that $3x^2 + 2 = 6$, then $x^2 + 7 = -10$ T

(iii) If two parallel lines intersect in exactly one point, then every 2017 lines intersect in exactly two points. T

QUESTION 15. (i) **(6 points)** Is the sequence 3, 2, 2, 2, 1 a graphical? If yes, then find the girth and the diameter of the graph. What is the adjacency matrix of the graph? Is the graph a bipartite? explain

$\underbrace{3\ 2\ 2\ 2\ 1}$ $\Rightarrow$ $\underline{1\ 1\ 1\ 1}$ $\rightarrow$ [ ]

$\boxed{\text{It is graphical}}$



girth = 3, diam = 3

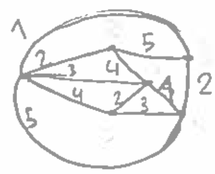
$$
\begin{array}{c|ccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & 0 & 1 & 0 & 0 & 0 \\
v_2 & 1 & 0 & 1 & 0 & 0 \\
v_3 & 0 & 1 & 0 & 1 & 1 \\
v_4 & 0 & 0 & 1 & 0 & 1 \\
v_5 & 0 & 0 & 1 & 1 & 0
\end{array}
$$

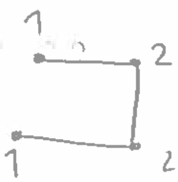
The graph is <u>not</u> Bipartite because it has an odd cycle

(ii) **(6 points)** Is the sequence 5, 4, 4, 3, 3, 3 a graphical? If yes, then find number of all edges of such graph and find its χ'

$\boxed{\text{It is graphical}}$

5 4 4 3 3 3 $\Rightarrow$ 3 3 2 2 2 $\Rightarrow$ $\underline{2\ 2\ 1\ 1}$ $\Rightarrow$





$\sum \deg(v) = 2|E|$

$5+4+4+3+3+3 = 2|E| \Rightarrow 22 = 2|E| \Rightarrow \boxed{|E| = 11}$

$\boxed{\chi' = 5}$

Faculty information

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